1. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

a) $\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx$

let $u = 1+\sqrt{x}$

$u - 1 = \sqrt{x}$

$x = (u-1)^2$

$\frac{dx}{du} = 2(u-1)$ 1

$\Rightarrow dx = 2(u-1)\,du$

when $x=0$, $u=1$

when $x=16$, $u=5$

$u = 1+\sqrt{0} = 1$

$u = 1+\sqrt{16} = 1+4 = 5$

$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_1^5 \frac{(u-1)^2}{u} \times 2(u-1)\,du = \int_1^5 \frac{2(u-1)^3}{u}\,du$ 1 1

b) $\int_1^5 \frac{2(u-1)^3}{u} \, du = 2\int_1^5 \frac{(u-1)^3}{u} \, du$

$= 2\int_1^5 \frac{u^3 - 3u^2 + 3u - 1}{u} \, du$

$= 2\int_1^5 u^2 - 3u + 3 - \frac{1}{u} \, du$ (1)

$= 2\left[\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln|u|\right]_1^5$ (1)

$= 2\left(\frac{1}{3}(5)^3 - \frac{3}{2}(5)^2 + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1\right)\right)$ (1)

$= \frac{104}{3} - 2\ln 5$ (1)

2. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$(3+x)^{-2}$$

writing each term in simplest form.

(4)

(b) Using the answer to part (a) and using algebraic integration, estimate the value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,\mathrm{d}x$$

giving your answer to 4 significant figures.

(4)

(c) Find, using algebraic integration, the exact value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,\mathrm{d}x$$

giving your answer in the form $a\ln b + c$, where a, b and c are constants to be found.

(5)

a) $(3+x)^{-2} = \left(3\left(1+\frac{x}{3}\right)\right)^{-2} = 3^{-2}\left(1+\frac{x}{3}\right)^{-2}$

1 1 1

$= \frac{1}{9}\left(1 - \frac{2x}{3} + \frac{(-2)(-2-1)}{2!}\left(\frac{x}{3}\right)^2 + \ldots\right)$

$= \frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ 1

b) $\int 6x(3+x)^{-2}\,dx \approx \int 6x\left(\frac{1}{9} - \frac{2x}{27} - \frac{x^2}{27}\right)dx$

$= \int\left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9}\right)dx$ 1

$$\int_{0.2}^{0.4}\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)dx=\left[\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}\right]_{0.2}^{0.4}$$ (1)

$$=\left(\frac{(0.4)^2}{3}-\frac{4(0.4)^3}{27}+\frac{(0.4)^4}{18}\right)-\left(\frac{(0.2)^2}{3}-\frac{4(0.2)^3}{27}+\frac{(0.2)^4}{18}\right)$$ (1)

$$=\frac{223}{6570}$$

$$=0.03304 \text{ (4sf)}$$ (1)

c) $\displaystyle\int_{0.2}^{0.4}\frac{6x}{(3+x)^2}\,dx$ let $u=3+x \Rightarrow x=u-3$

$du=dx$

limits: $(0.2, 0.4)\rightarrow(3.2, 3.4)$

$$=\int_{3.2}^{3.4}\frac{6(u-3)}{u^2}\,du=\int_{3.2}^{3.4}\left(\frac{6}{u}-18u^{-2}\right)du$$ (1)

$$=\Big[6\ln|u| + 18u^{-1}\Big]_{3.2}^{3.4}$$ (1) (1)

$$=\left(6\ln 3.4+\frac{18}{3.4}\right)-\left(6\ln 3.2+\frac{18}{3.2}\right)=6\ln\left(\frac{17}{16}\right)-\frac{45}{136}$$ (1) (1)